THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 11

§81) 2) b) Since $f(z) = \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\operatorname{Log} z/(z+i)^2}{(z-i)^2}$ has a pole of order two at $z = i$, we have

$$\operatorname{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \left[\frac{d}{dz}\frac{\operatorname{Log} z}{(z+i)^2}\right]_{z=i} = \left[\frac{(z+i)^2\left(\frac{1}{z}\right) - (\operatorname{Log} z)(2)(z+i)}{(z+i)^4}\right]_{z=i} = \frac{\pi + 2i}{8}$$

§81) 3) a) Note that for $z \neq 0$, we have

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \dots$$

Therefore, $z = 0$ is a pole of order $m = 3$ with residue $B = \dfrac{1}{3!} = \dfrac{1}{6}$.

§81) 4) Note that the singularities of the integrand are $z = 1$ and $z = \pm 3i$.

a) Since $z = 1$ is the only singular point lying inside the contour C, by Cauchy's residue Theorem, we have

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \operatorname{Res}_{z=1}\left[\frac{(3z^3+2)/(z^2+9)}{z-1}\right] = 2\pi i\left(\frac{3(1)^3+2}{1^2+9}\right) = \pi i$$

b) Since all the singular points lies inside the contour C, by Cauchy's residue Theorem, we have

$$\begin{aligned}
&\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz \\
&= 2\pi i\left[\operatorname{Res}_{z=1}\left[\frac{(3z^3+2)/(z^2+9)}{z-1}\right] + \operatorname{Res}_{z=3i}\left[\frac{(3z^3+2)/(z-1)(z+3i))}{z-3i}\right]\right. \\
&\qquad \left. + \operatorname{Res}_{z=-3i}\left[\frac{(3z^3+2)/(z-1)(z-3i)}{z+3i}\right]\right] \\
&= 6\pi i
\end{aligned}$$

§81) 7) a) Note that the singularities of the function $f(z)$ are $z = 0, 1$ and $-\dfrac{5}{2}$. All of them lie inside the contour $|z| = 3$.

Furthermore, $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{(3+2z)^2/(1-z)(2+5z)}{z}$.

As a result,

$$\int_{|z|=3} f(z)dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i \frac{(3+2(0))^2}{(1-0)(2+5(0))} = 9\pi i$$

§83) 3) a) Note that $\cosh(\pi i/2) = 0$ and $\cosh'(\pi i/2) = \sinh(\pi i/2) \neq 0$. Therefore, $z = \pi i/2$ is a simple pole and the residue is given by

$$\operatorname{Res}_{z=\pi i/2} \frac{\sinh z/z^2}{\cosh z} = \left[\frac{\sinh z/z^2}{\sinh z}\right]_{z=\pi i/2} = -\frac{4}{\pi^2}$$

§83) 4) b) Recall that $\tanh z = \dfrac{\sinh z}{\cosh z}$. Since $\cosh(z_n) = 0$ and $\cosh'(z_n) = \sinh(z_n) \neq 0$, $z = z_n$ is a simple pole and the residue is given by

$$\text{Res}_{z=z_n} \frac{\sinh z}{\cosh z} = \left[\frac{\sinh z}{\sinh z}\right]_{z=z_n} = 1$$

§86) 3) Note that $z^4+1 = 0$ if and only if $z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}$ or $e^{i7\pi/4}$. Let C_R be the upper semicircle centered at origin with radius R in positive orientation. By Cauchy's Residue Theorem, for R large enough we have

$$\int_{-R}^{R} \frac{1}{x^4+1}dx + \int_{C_R} \frac{1}{z^4+1}dz = 2\pi i \left[\text{Res}_{z=e^{i\pi/4}}\left(\frac{1}{z^4+1}\right) + \text{Res}_{z=e^{i3\pi/4}}\left(\frac{1}{z^4+1}\right)\right]$$

Note that

$$\text{Res}_{z=e^{i\pi/4}}\left(\frac{1}{z^4+1}\right) = \left[\frac{1}{4z^3}\right]_{z=e^{i\pi/4}} = \frac{-e^{i\pi/4}}{4}$$
$$\text{Res}_{z=e^{i3\pi/4}}\left(\frac{1}{z^4+1}\right) = \left[\frac{1}{4z^3}\right]_{z=e^{i3\pi/4}} = \frac{-e^{i3\pi/4}}{4}$$

Furthermore, since

$$\left|\frac{1}{z^4+1}\right| \leq \frac{1}{|z|^4-1} = \frac{1}{R^4-1},$$

we have

$$\left|\int_{C_R} \frac{1}{z^4+1}dz\right| \leq \frac{1}{R^4-1} \times \pi R \to 0$$

as $R \to \infty$.

As a result, we have

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = 2\pi i(-\frac{e^{i\pi/4}}{4} - \frac{e^{i3\pi/4}}{4}) = \frac{\pi}{\sqrt{2}}$$

Since $\dfrac{1}{x^4+1}$ is an even function, we have

$$\int_{0}^{\infty} \frac{1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}$$

§86) 8) Note that $(z^2+1)(z^2+2z+2) = 0$ if and only if $z = i, -i, -1+i$ or $-1-i$. Let C_R be the upper semicircle centered at origin with radius R in positive orientation. By Cauchy's Residue Theorem, for R large enough we have

$$\int_{-R}^{R} \frac{x}{(x^2+1)(x^2+2x+2)}dx + \int_{C_R} \frac{z}{(z^2+1)(z^2+2z+2)}dz$$
$$= 2\pi i \left[\text{Res}_{z=i}\left(\frac{z}{(z^2+1)(z^2+2z+2)}\right) + \text{Res}_{z=-1+i}\left(\frac{z}{(z^2+1)(z^2+2z+2)}\right)\right]$$

Note that

$$\text{Res}_{z=i}\left(\frac{z}{(z^2+1)(z^2+2z+2)}\right) = \left[\frac{z}{(z+i)(z^2+2z+2)}\right]_{z=i} = \frac{1}{2(1+2i)}$$

$$\text{Res}_{z=-1+i}\left(\frac{z}{(z^2+1)(z^2+2z+2)}\right) = \left[\frac{z}{(z^2+1)(z+1+i)}\right]_{z=-1+i} = \frac{1+i}{2(1-2i)}$$

Furthermore, since

$$\left|\frac{z}{(z^2+1)(z^2+2z+2)}\right| \le \frac{|z|}{(|z|^2-1)(|z|^2-2|z|-2)} = \frac{R}{(R^2-1)(R^2-2R-1)},$$

we have

$$\left|\int_{C_R} \frac{z}{(z^2+1)(z^2+2z+2)} dz\right| \le \frac{R}{(R^2-1)(R^2-2R-1)} \times \pi R \to 0$$

as $R \to \infty$.

As a result, we have

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)} dx = 2\pi i \left(\frac{1}{2(1+2i)} + \frac{1+i}{2(1-2i)}\right) = -\frac{\pi}{5}$$

§88) 5) Note that $z^4+4=0$ if and only if $z=\sqrt{2}e^{i\pi/4}, \sqrt{2}e^{i3\pi/4}, \sqrt{2}e^{i5\pi/4}$ or $\sqrt{2}e^{i7\pi/4}$. Let C_R be the upper semicircle centered at origin with radius R in positive orientation. By Cauchy's Residue Theorem, for R large enough we have

$$\int_{-R}^{R} \frac{x^3e^{iax}}{x^4+4} dx + \int_{C_R} \frac{z^3e^{iaz}}{z^4+4} dz = 2\pi i \left[\text{Res}_{z=\sqrt{2}e^{i\pi/4}}\left(\frac{z^3e^{iaz}}{z^4+4}\right) + \text{Res}_{z=\sqrt{2}e^{i3\pi/4}}\left(\frac{z^3e^{iaz}}{z^4+4}\right)\right]$$

Note that

$$\text{Res}_{z=\sqrt{2}e^{i\pi/4}}\left(\frac{z^3e^{iaz}}{z^4+4}\right) = \left[\frac{z^3e^{iaz}}{4z^3}\right]_{z=\sqrt{2}e^{i\pi/4}} = \frac{e^{-a+ia}}{4}$$

$$\text{Res}_{z=\sqrt{2}e^{i3\pi/4}}\left(\frac{z^3e^{iaz}}{z^4+4}\right) = \left[\frac{z^3e^{iaz}}{4z^3}\right]_{z=\sqrt{2}e^{i3\pi/4}} = \frac{e^{-a-ia}}{4}$$

Furthermore, since

$$\left|\frac{z^3}{z^4+4}\right| \le \frac{|z|^3}{(|z|^4-4)} = \frac{R^3}{(R^4-4)} \to 0 \text{ as } R \to \infty$$

by Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{C_R} \frac{z^3e^{iaz}}{z^4+4} dz = 0.$$

As a result, we have

$$\int_{-\infty}^{\infty} \frac{x^3e^{iax}}{x^4+4} dx = 2\pi i \left(\frac{e^{-a+ia}}{4} + \frac{e^{-a-ia}}{4}\right) = i\pi e^{-a}\cos a$$

Hence

$$\int_{-\infty}^{\infty} \frac{x^3\sin(ax)}{x^4+4} dx = \text{Im} \int_{-\infty}^{\infty} \frac{x^3e^{iax}}{x^4+4} dx = \pi e^{-a}\cos a$$

§88) 7) Note that $(z^2+1)(z^2+9)=0$ if and only if $z=i,-i,3i$ or $-3i$. Let C_R be the upper semicircle centered at origin with radius R in positive orientation. By Cauchy's Residue Theorem, for R large enough we have

$$\int_{-R}^{R} \frac{x^3e^{ix}}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{z^3e^{iz}}{(z^2+1)(z^2+9)} dz$$
$$= 2\pi i \left[\text{Res}_{z=i}\left(\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}\right) + \text{Res}_{z=3i}\left(\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}\right)\right]$$

Note that

$$\text{Res}_{z=i}\left(\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}\right) = \left[\frac{z^3e^{iz}}{(z+i)(z^2+9)}\right]_{z=i} = \frac{-e^{-1}}{16}$$

$$\text{Res}_{z=3i}\left(\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}\right)=\left[\frac{z^3e^{iz}}{(z^2+1)(z+3i)}\right]_{z=3i}=\frac{9e^{-3}}{16}$$

Furthermore, since

$$\left|\frac{z^3}{(z^2+1)(z^2+9)}\right|\leq\frac{|z|^3}{(|z|^2-1)(|z|^2-9)}=\frac{R^3}{(R^2-1)(R^2-9)}\to 0 \text{ as } R\to\infty$$

by Jordan's Lemma we have

$$\lim_{R\to\infty}\int_{C_R}\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}dz=0.$$

As a result, we have

$$\int_{-\infty}^{\infty}\frac{x^3e^{ix}}{(x^2+1)(x^2+9)}dx=2\pi i\left(\frac{-e^{-1}}{16}+\frac{9e^{-3}}{16}\right)=i\frac{\pi}{8}(9e^{-3}-e^{-1})$$

Hence

$$\int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)(x^2+9)}dx=\text{Im}\int_{-\infty}^{\infty}\frac{x^3e^{ix}}{(x^2+1)(x^2+9)}dx=\frac{\pi}{8}(9e^{-3}-e^{-1})$$

Since the integrand is an even function, we have

$$\int_{0}^{\infty}\frac{x^3\sin x}{(x^2+1)(x^2+9)}dx=\frac{\pi}{16}(9e^{-3}-e^{-1})$$